Your Signature ______________________

Instructions:
1. For writing your answers use both sides of the paper in the answer booklet.

2. Please write your name on every page of this booklet and every additional sheet taken.

3. If you are using a Theorem/Result from class please state and verify the hypotheses of the same.

4. **Maximum time is 3 hours and Maximum Possible Score is 100.**

Score

Q.No.	**Alloted Score**	**Score**
1.	17	
2.	17	
3.	17	
4.	17	
5.	17	
6.	17	
Total	105	

Number of Extra sheets attached to the answer script: ______________________

1. Let $\{Y_n\}_{n\geq 1}$ be a sequence of bounded random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

$$Y = \liminf_{n\to\infty} Y_n$$

is measurable.

2. Let $\{X_n\}_{n\geq 1}$ be independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose $X_n \sim \text{Normal}(0,1)$ then show that $\mathbb{P}(\limsup_{n\to\infty} \frac{X_n}{\sqrt{2\log(n)}} = 1) = 1$.

3. Let $\{X_n\}_{n\geq 1}$ be non-negative i.i.d. random variables.

   (a) Is $\lim\limits_{n\to\infty} \frac{X_1+X_2+...+X_n}{n}$ measurable w.r.t the tail $\sigma-$algebra ?

   (b) Is the event $\{\lim\limits_{n\to\infty} \{\sum_{i=1}^{n} X_i = 5\}$ always in the tail $\sigma-$algebra ?

4. Let Z_n be i.i.d random variables on $(\Omega, \mathcal{F}, \mathbb{P})$such that

$$\mathbb{P}(Z_n = 1) = \frac{1}{2} = 1 - \mathbb{P}(Z_n = 0).$$

Define $X_n = \frac{Z_n}{n^\theta}$ for $0 < \theta$. Decide whether the series with partial sums $S_n = \sum_{j=1}^{n} X_n$ converges almost surely or not ?

5. Suppose $\{X_n\}_{n\geq 1}, X$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$E[X_n^2] < \infty, \forall n \geq 1, \quad \text{and} \quad E[(X_n - X)^2] \to 0 \text{ as } n \to \infty.$$

Show that $\{X_n^2\}_{n\geq 1}$ is uniformly integrable.

6. Let $\mathbb{Q}$ be the distribution of X on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Show that for any Borel-measurable function $f : \mathbb{R} \to \mathbb{R}$

$$\int_{\mathbb{R}} f(y) d\mathbb{Q}(y) = \int_{\Omega} f(X(\omega)) d\mathbb{P}(\omega),$$

assuming both sides exists.